Worksheet 18.1 - Solutions

Section 18.1 Summary

- State Green's thm:

$$\oint_{\partial D} F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

or

$$\oint_{\partial D} F\cdot dr = \iint_D \text{curl}_z(F)\,dA$$

- Formulas for the area of the region D enclosed by C: $\text{Area}(D) = \oint_C x\,dy = \oint_C -y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$

- Vector form of Green's thm:

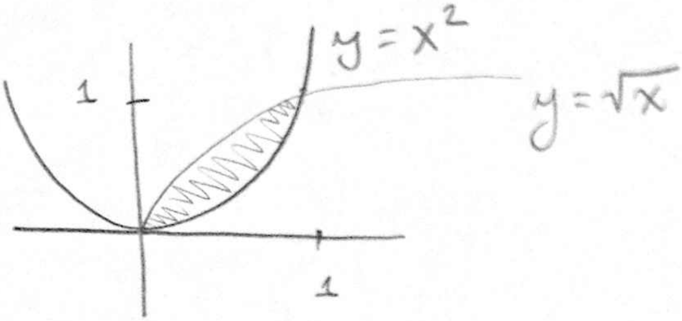
$$\oint_{\partial D} F\cdot n\,ds = \iint_D \text{div}(F)\,dA$$

1. (a) Let D be the region enclosed by C. Then, Green's thm tells us that

$$\oint_C F \cdot dr = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$= \iint_D (2x - 1)\, dA$$

Sketch D:



So, the integral becomes,

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (2x-1)\, dy\, dx = \int_0^1 (2x-1)y \Big|_{x^2}^{\sqrt{x}} dx$$

$$\int_0^1 2x^{3/2} - x^{1/2} - 2x^3 + x^2\, dx = \frac{2 \cdot 2}{5} x^{5/2} - \frac{2x^{3/2}}{3} - \frac{x^4}{2} + \frac{x^3}{3} \Big|_0^1$$

$$= \frac{4}{5} - \frac{2}{3} - \frac{1}{2} + \frac{1}{3} = \frac{4}{5} - \frac{1}{3} - \frac{1}{2} = \boxed{-\frac{1}{30}}$$

(b) $F(x,y) = \langle x^2, x^2 \rangle$ and C consists of the arcs $y=x^2$ and $y=x$ for $0 \le x \le 1$.

$$\oint_C F \cdot dr = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_D 2x \, dA$$

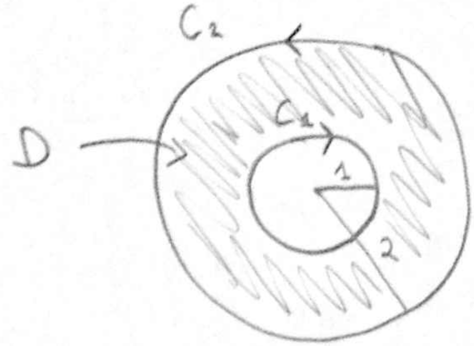
D: region enclosed by C

$$= \int_0^1 \int_{x^2}^{x} 2x \, dy\,dx = \int_0^1 2xy \Big|_{x^2}^{x} dx = \int_0^1 2x^2 - 2x^3 \, dx$$

$$= \frac{2x^3}{3} - \frac{2x^4}{4} \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$$

2. Let C_R be the circle of radius R centered at the origin. Use the general form of Green's thm to determine $\oint_{C_2} F \cdot dr$, where $\oint_{C_1} F \cdot dr = 9$ and $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x^2 + y^2$ in the annulus $1 \le x^2 + y^2 \le 4$.

$\hookrightarrow$



By Green's thm:

$$\oint_{\partial D} F\cdot dr = \oint_{\partial C_2} F\cdot dr - \oint_{\partial C_1} F\cdot dr$$

$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \Rightarrow$$

$$\oint_{\partial C_2} F\cdot dr = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA + \iint_{\partial C_1} F\cdot dr$$

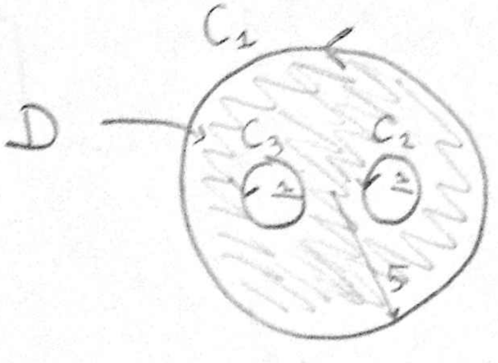
$$= \iint_D x^2+y^2 \, dA + 9$$

$$= \int_0^{2\pi}\int_1^2 r^3 \, dr\, d\theta + 9$$

$$= 2\pi \left(\frac{r^4}{4}\Big|_1^2\right) + 9$$

$$= 2\pi \cdot 4 - \frac{2\pi}{4} + 9 = 8\pi - \frac{\pi}{2} + 9 = \boxed{\frac{15\pi}{2} + 9}$$

3.



Suppose that

$$\oint_{C_2} F \cdot dr = 3\pi \quad \text{and} \quad \oint_{C_3} F \cdot dr = 4\pi$$

Find $\oint_{C_1} F \cdot dr$ assuming that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 9$ in D.

We have

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_{\partial D} F \cdot dr = \oint_{C_1} F \cdot dr - \oint_{C_3} F \cdot dr - \oint_{C_2} F \cdot dr$$

$$\Longrightarrow$$

$$\oint_{C_1} F \cdot dr = \iint_D 9 \, dA + \oint_{C_3} F \cdot dr + \oint_{C_2} F \cdot dr$$

$$= 9 \cdot \text{area}(D) + 7\pi$$

$$= 9\left[\pi 5^2 - 2 \cdot \pi\right] + 7\pi$$

$$\boxed{= 214\pi}$$

4. Let C_R be the circle of radius R centered at the origin. Use Green's thm to find the value that maximizes $\oint_{C_R} y^3 dx + x dy$.

Fix $R>0$, then by Green's thm,

$$\oint_{C_R} \underbrace{y^3}_{F_1} dx + \underbrace{x}_{F_2} dy = \iint_{D_R} (1 - 3y^2) \, dA$$

D_R ← region enclosed by C_R

$$= \int_0^{2\pi}\int_0^R (1 - 3r^2 \sin^2\theta) r \, dr d\theta$$

$$= \int_0^{2\pi} \frac{R^2}{2} - \frac{3R^4}{4}\sin^2\theta \, d\theta = R^2\pi - \frac{3R^4}{4}\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= R^2\pi - \frac{3R^4}{4}\int_0^{2\pi} 1 - \cos 2\theta \, d\theta = R^2\pi - \frac{3R^4}{4}\left[2\pi\right] = R^2\pi - \frac{3R^4}{4}\pi$$

$$= \pi\left(R^2 - \frac{3R^4}{4}\right).$$

↪

So, we want to maximize $G(R) = \left(R^2 - \frac{3R^4}{4}\right)$.

$$G'(R) = 2R - 3R^3 = 0 \Rightarrow$$

$$R = 0 \quad \text{or} \quad 2 - 3R^2 = 0 \Rightarrow$$

$$R = \pm\sqrt{2/3}$$

So, it is maximized when $R = \sqrt{2/3}$, $R=0$ is clearly a minimum.

5. $F = \langle 2x + y^3, 3y - x^4 \rangle$ across the boundary of the unit circle.

By the vector for of green's thm, we have,

$$\text{Flux}(F) = \int_{\partial D} F \cdot n \, dr = \iint_D \operatorname{div}(F) \, dA = \iint_D (2+3) \, dA$$

$$= 5 \cdot \text{area}(D) = \boxed{5\pi}$$

6. $F = \langle xy^2 + 2x,\ x^2y - 2y \rangle$ across the boundary of the region described by $x^2 + y^2 \le 3$, $y \ge 0$. (semi circle radius $\sqrt{3}$)

$$\text{Flux}(F) = \oint_{\partial D} (F \cdot n)\, dr = \iint_D (y^2 + 2 + x^2 - 2)\, dA$$

$$= \int_0^{\pi} \int_0^{\sqrt{3}} r^2 \cdot r \, dr\, d\theta$$

$$= \pi \int_0^{\sqrt{3}} r^3 \, dr = \pi \left[\frac{r^4}{4} \Big|_0^{\sqrt{3}} \right] = \boxed{\frac{9\pi}{4}}$$

7. Let F be a velocity field. Estimate the circulation of F around a circle of radius $R = 0.05$ with center P, assuming that $\text{curl}_z(F)(P) = -3$. Which direction would a paddle placed at P spin?

$$\oint_C F \cdot ds = \iint_D \text{curl}_z(F)\, dA \approx -3 \iint_D dA = -3 \cdot \pi (0.05)^2$$
$$= -0.024$$

Since the curl is neg. it would spin in the clockwise direction.